# Nonparametric Bayesian volatility learning under microstructure noise

Shota Gugushvili[♡], Frank van der Meulen[◇], Moritz Schauer[♣] and Peter Spreij[♠]

Cramér-Meeting September 2020

[♡]Wageningen University, [◇]Delft University of Technology, [♣]Chalmers and University of Gothenburg, and [♠]University of Amsterdam

# Team

# Contents


Problem formulation

Learning procedure

Numerical experiment

Application to real data

Summary

# Problem formulation

# Problem

- Model: Stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + s(t)\mathrm{d}W_t, \quad X_0 = x_0, \quad t \in [0, T].$$

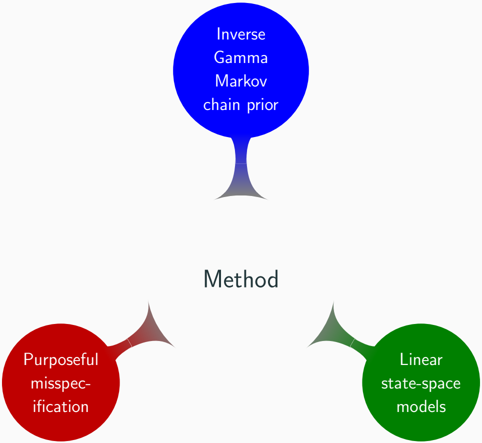
- Task: Learn the *volatility function* $s$ from noisy observations on $X$ at times $0 < t_1 < \ldots < t_n = T$:

$$Y_i = X_{t_i} + V_i.$$

  The $V$'s are unobserved stochastic disturbances.

- Important problem in *finance*; data subject to *microstructure noise* and not on a *uniform* time grid.

# Sources of inspiration



# Method

- *Minimalistic* in its assumptions on the volatility function, which in particular can be a stochastic process.
- Intuitive to understand. Ingredients are well-known techniques.
- Posterior inference via Gibbs sampler.

# Learning procedure

# Main challenge

- *Closed form expression* for the posterior distribution not available; numerically intractable likelihood.
- Remedy: misspecification on purpose
- Alternative: Data augmentation device and some carefully chosen MCMC sampler.

# Misspecification on purpose

- Model misspecification not necessarily bad, if you know what you are doing; cf. GUGUSHVILI ET AL. '20 and MARTIN ET AL. '18.
- *Misspecify the model purposely* and act as if the drift $b = 0$. Ultimate justification: Girsanov's theorem.
- Then

$$X_{t_i} = X_{t_{i-1}} + U_i.$$

Here

$$U_i = \int_{t_{i-1}}^{t_i} s(t)\mathrm{d}W_t \sim N(0, w_i),$$

and $w_i = \int_{t_{i-1}}^{t_i} s^2(t)\mathrm{d}t$. Note that $\{U_i\}$ are independent.

# Gaussian linear state-space model

- Simplify the notation: $x_i = X_{t_i}$, $y_i = Y_i$, $u_i = U_i$, $v_i = V_i$. Then

$$x_i = x_{i-1} + u_i,$$
$$y_i = x_i + v_i.$$

  This is a *linear state-space model*.
- Assume that $\{v_i\} \overset{iid}{\sim} N(0, \eta_v)$ are independent of the Wiener process $W$, so that $\{v_i\}$ are independent of $\{u_i\}$.
- The model becomes *Gaussian*. This is very convenient.
- Notation:

$$\mathcal{X}_n = \{x_i\}, \quad \mathcal{Y}_n = \{y_i\}.$$

# Histogram prior for volatility

- Define $N$ bins

$$B_k = [t_{m(k-1)}, t_{mk}), \quad k = 1, \dots, N.$$

where $n = mN$.

- Assume piecewise constant (squared) volatility,

$$s = \sum_{k=1}^{N} \sqrt{\theta_k} \mathbf{1}_{B_k}, \qquad s^2 = \sum_{k=1}^{N} \theta_k \mathbf{1}_{B_k}.$$

- Assume $\{\theta_k\}$ form an inverse Gamma Markov chain...

## Inverse Gamma Markov chain

- Fix hyperparameters $\alpha_1$, $\alpha$ and define a Markov chain on $\theta_1, \zeta_2, \theta_2, \ldots, \zeta_k, \theta_k, \ldots, \zeta_N, \theta_N$ with (latent) auxiliary variables $\zeta_k$, $k = 2, \ldots, N$, and transition densities specified via

$$\begin{aligned} \theta_1 &\sim \mathrm{IG}(\alpha_1, \alpha_1), \\ \zeta_{k+1} \mid \theta_k &\sim \mathrm{IG}(\alpha, \alpha\theta_k^{-1}), \\ \theta_{k+1} \mid \zeta_{k+1} &\sim \mathrm{IG}(\alpha, \alpha\zeta_{k+1}^{-1}). \end{aligned}$$

- *Mild smoothing/regularisation* through the prior-induced positive correlation of the volatility function across close bins.

# Realisations of volatility under the prior

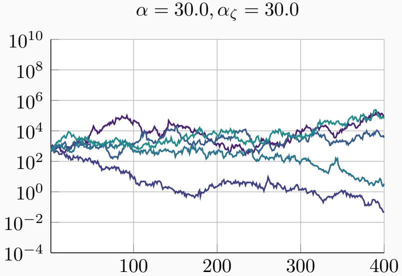



**Figure 1:** Realisations of $\{\theta_k\}$ with $\alpha = 30$ for $N = 400$. If $\log \theta_k$ looks like $cW_t$ to you... you are right.

# Realisations of volatility under the prior

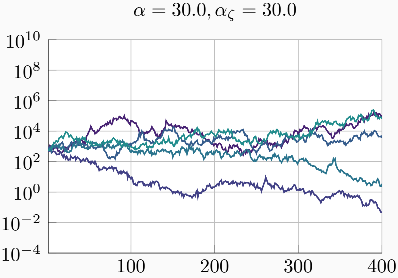



**Figure 1:** Realisations of $\{\theta_k\}$ with $\alpha = 30$ for $N = 400$.

# Realisations of volatility under the prior

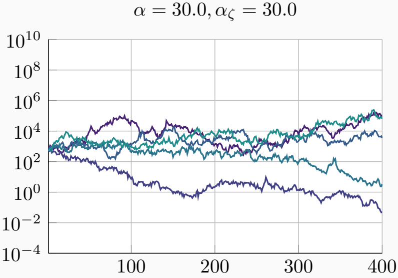



**Figure 1:** Realisations of $\{\theta_k\}$ with $\alpha = 30$ for $N = 400$. If $\log \theta_k$ looks like $cW_t$ to you... you are right.

# Prior

- Drift $b \equiv 0$.
- IGMC for $\{\theta_i\}$.
- Diffuse prior for $\alpha$; improper for $\alpha_1$.
- Noise variance $\eta_v \sim \mathrm{IG}$.

# Gibbs sampler

- Sequential updates:

$$
\begin{aligned}
\zeta_2, \ldots, \zeta_N \mid \mathcal{X}_n,\ \alpha, \theta_1, \ldots, \theta_N &\quad (\text{conjugate}) \\
\theta_1, \ldots, \theta_N \mid \mathcal{X}_n, \alpha, \zeta_2, \ldots, \zeta_N &\quad (\text{conjugate}) \\
\alpha \mid \theta_1, \zeta_2, \ldots, \zeta_N, \theta_N &\quad (\text{generic MH}) \\
\mathcal{X}_n \mid \mathcal{Y}_n, \eta_v, \theta_1, \ldots, \theta_N &\quad (\text{Kalman-based}) \\
\eta_v \mid \mathcal{V}_n &\quad (\text{conjugate})
\end{aligned}
$$

# Numerical experiment

# Heston model i

- Widely used stochastic volatility model (Heston '93).
- Price process $S$ evolving according to the SDE

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sqrt{Z_t} S_t \mathrm{d}W_t,$$

where the process $Z$ follows the CIR (Cox et al. '85) or square root process,

$$\mathrm{d}Z_t = \kappa(\theta - Z_t)\mathrm{d}t + \sigma\sqrt{Z_t}\mathrm{d}B_t.$$

$W$ and $B$ are correlated Wiener processes with correlation $\rho$.

# Heston model ii

- By Itō's formula, logarithm $X_t = \log S_t$ obeys the diffusion equation with volatility
$$s(t) = \sqrt{Z_t}.$$
- In the Heston model the volatility function $s$ is random, with its law not known in closed form.

# Heston model iii

- High frequency observations on the log-price process $X$ with additive noise $V_i \sim N(0, \eta)$. No further knowledge of the data generation mechanism.
- Simulated path using "realistic" values (cf. Table 1 in Heston '93)

$$\mu = 0.05, \quad \kappa = 7, \quad \theta = 0.04, \quad \sigma = 0.6, \quad \rho = -0.6.$$

- Noise variance $\eta_v = 10^{-6}$ yielding a meaningful signal-to-noise ratio.

# Heston model iv

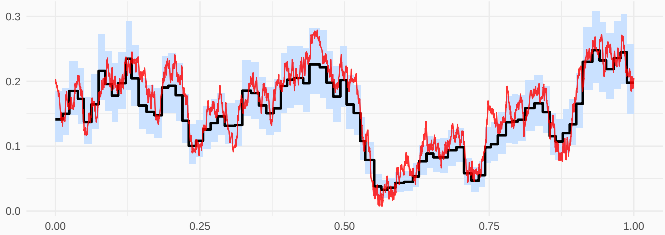



**Figure 2:** Posterior mean and pointwise $95\%$ credible band for $N = 80$ bins. True volatility function plotted in red; the black step function is the posterior mean.

# Application to real data

# Exchange rate data i

- Volatility learning from the high frequency exchange rate data.
- EUR/USD tick data (bid prices) for 2 March 2015, subsampled by retaining every $10$th observation. A total of $n = 13\,025$ observations (subsampling accounts for the assumption of independent additive measurement noise).
- Log-transform the observed time series and assume the additive measurement error model.

# Exchange rate data ii

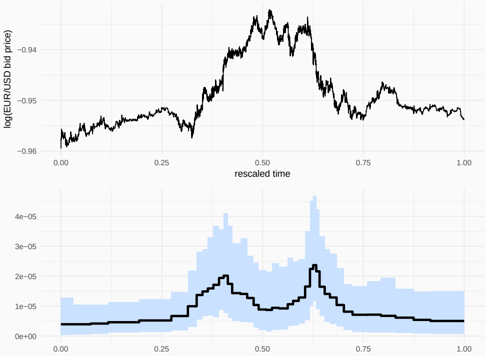



**Figure 3:** Top: Logarithm of the EUR/USD exchange rate data for 2 March 2015. Bottom: Posterior mean (black curve) and pointwise 95% credible band (blue band) for the volatility function.

# Summary

# Key points

- A conceptually straightforward, clever nonparametric Bayesian method for volatility learning under microstructure noise.
- Available as julia package; see Schauer and Gugushvili '20.
- Partial fulfillment of prediction in Godsill et al. '07 that some ideas developed originally in the context of audio and music processing "*will also find use in other areas of science and engineering, such as financial or biomedical data analysis*".

# References i


- A.T. Cemgil and O. Dikmen. Conjugate gamma Markov random fields for modelling nonstationary sources. *7th International Conference on Independent Component Analysis and Signal Separation (ICA 2007)*, pages 697–705.
- J. C. Cox, J. E. Ingersoll and S. A. Ross (1995). A theory of the term structure of interest rates. *Econometrica*, 53:385–407.
- S. L. Heston (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financial Stud.*, 6:327–343.

- S. J. Godsill, A. T. Cemgil, C. Févotte and P. J. Wolfe (2007). Bayesian computational methods for sparse audio and music processing, *15th European Signal Processing Conference (EURASIP)*, pages 345–349.
- S. Gugushvili, F. van der Meulen, M. Schauer and P. Spreij (20xx). Nonparametric Bayesian estimation of a Hölder continuous diffusion coefficient. To appear in *Braz. J. Probab. Stat.*
- S. Gugushvili, F. van der Meulen, M. Schauer and P. Spreij (2018). Nonparametric Bayesian volatility learning under microstructure noise. arXiv:1805.05606 [stat.ME].

# References iii

R. Martin, C. Ouyang and F. Domagni (2018). 'Purposely misspecified' posterior inference on the volatility of a jump diffusion process. *Stat. Probab. Lett.*, 134:106–113.

M. Schauer and S. Gugushvili. (2020). MicrostructureNoise.jl: v0.11.0. *Zenodo*. DOI: 10.5281/zenodo.1241010.