Fourier method for valuation of options under parameter and state uncertainty

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Outline

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Pension problem - redistribution of income over time

Mix of assets limits the market risk while protecting against inflation, cf. Mulvey and Holen [2016].

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We have worked in two directions

- Portfolio optimization
  - Drawdown control, Nystrup et al. [2019]
  - Hyperparameter optimization, Nystrup et al. [2020]
- Option pricing - replication of insurance contract, e.g. Lindström [2019]

Information structures

Theory

Definition 16.6 (Filtrations and adaptedness). A **filtration** is a family $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,\infty)}$ of sub-σ-algebras of $\mathcal{F}$ with the property that $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $0 \le s < t$. A stochastic process $X = \{X_t\}_{t\in[0,\infty)}$ is said to be **adapted** to $\mathbb{F}$ if $X_t \in \mathcal{F}_t$, for all t.

The **natural filtration** $\mathbb{F}^X$ of a stochastic process $\{X_t\}_{t\in[0,\infty)}$ is the smallest filtration with respect to which X is adapted, i.e.,

$$\mathcal{F}_t^X = \sigma(X_s; s \le t),$$

and is related to the situation where the available information at time t is obtained by observing the values of the process (and nothing else) X up to time t.

Practice

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Practice

In theory there is no difference between theory and practice. In practice there is.

Albert Einstein

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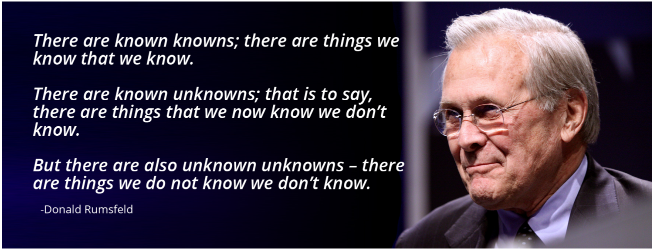
Risk and uncertainty



Former US Secretary of Defense, Donald Rumsfeld in the context of dealing with terrorism

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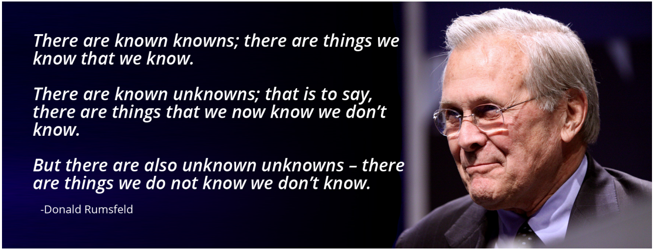

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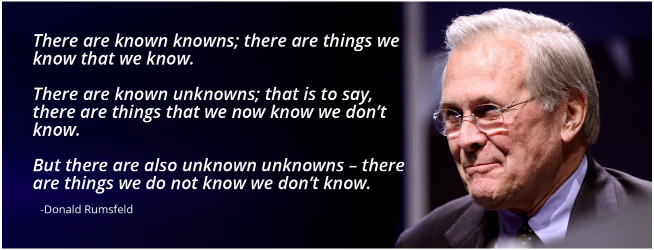

Former US Secretary of Defense, Donald Rumsfeld in the context of dealing with terrorism

- Extensive theory on the difference between risk and uncertainty, cf. Knight [1921].
- Ellsberg [1961] showed that agents prefer risks to uncertainty, even when the risky outcome could be a loss while the uncertain is a guaranteed gain!

Information structures, revisited

Consider a model with observable processes S and latent processes V, with $Z_t = [S_t,\ V_t]$.

Model filtration

It is the smallest σ-algebra generated by the vector-valued process Z_t augmented by the $\mathbb{P}$-null sets $\mathcal{N}$,

$$\mathcal{F}_t = \sigma(Z_u, u \leq t) \vee \mathcal{N} \tag{1}$$

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Risk Neutral valuation

The arbitrage free price is given by the risk-neutral expectation

$$\pi_t(S_t|\theta) = p(t, T)\mathbf{E}^{\mathbb{Q}}\left[\Phi(S_T)|\mathcal{F}_t\right] \tag{2}$$

where $\Phi(S_T)$ is the *contact function* e.g.
$\Phi(S_T) = \max(S_T - K, 0)$

Information structures, revisited

Market filtration

It is the filtration generated by discrete observations (prices are recorded at $t_1, \ldots, t_k \leq t$) of the traded assets $S_{t_k}^{Market}$ and possibly also derivatives $\pi_{t_k}^{Market}(S_{t_k})$ written on those assets, augmented by the $\mathbb{P}$-null sets $\mathcal{N}$,

$$\mathcal{F}_t^{Obs} = \sigma\left(\left[S_{t_k}^{Market},\ \pi_{t_k}^{Market}(S_{t_k})\right],\ \forall t_k \leq t\right) \vee \mathcal{N} \quad (3)$$

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Note $\mathcal{F}_t^{Obs} \subset \mathcal{F}_t$

Implications

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Lindström [2010] suggested to use

$$\begin{aligned}\tilde{\pi}_t(S_t) &= p(t,T)\mathbf{E}^{\mathbb{Q}}\left[\Phi(S_T)|\mathcal{F}_t^{Obs}\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[p(t,T)\mathbf{E}^{\mathbb{Q}}\left[\Phi(S_T)|\mathcal{F}_t\right]|\mathcal{F}_t^{Obs}\right] \quad (4)\\ &= \mathbf{E}^{\mathbb{Q}}\left[\pi_t(S_t|\theta)|\mathcal{F}_t^{Obs}\right] = \int \pi_t(S_t|\theta)q(\theta)\mathrm{d}\theta \quad (5)\end{aligned}$$

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&= \mathbf{E}^{\mathbb{Q}}\left[\pi_t(S_t|\theta)|\mathcal{F}_t^{Obs}\right] = \int \pi_t(S_t|\theta)q(\theta)\mathrm{d}\theta \qquad (5)
\end{aligned}$$

Approximated by Monte Carlo in Lindström [2010].

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Fourier pricing

Carr and Madan [1999] showed that the price of a call option is given by $c^{\theta}(T,k) = e^{\alpha k} C^{\theta}(T,k), \quad \alpha > 0$

$$C^{\theta}(T,k) = \frac{e^{-\alpha k}}{\pi} \Re \left[\int_0^{\infty} g(u) \varphi_T^{\theta}(u - (\alpha+1)i) \right] \mathrm{d}u, \quad (6)$$

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where

$$g(u) = \frac{e^{-iuk} e^{-rT}}{\alpha^2 + \alpha - u^2 + iu(2\alpha + 1)}, \tag{7}$$

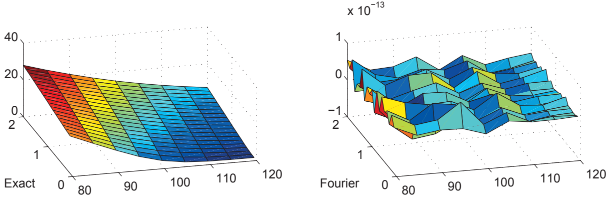
and

$$\varphi_T^{\theta}(u) = \mathbf{E}^{\mathbb{Q}}[e^{ius_T}] = \int_{-\infty}^{\infty} e^{ius_T} q_{\theta}(s_T | s_0) \mathrm{d}s_T \tag{8}$$

is the characteristic function.

Computational considerations

Adaptive Fourier-Gauss-Laguerre developed in Lindström et al. [2008] was shown in von Sydow et al. [2015] to be faster and more accurate than any other method for ”standard” problems.



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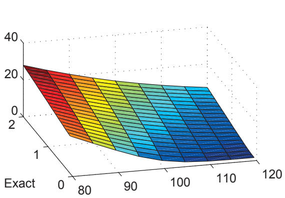

Still, the Monte Carlo method in Lindström [2010] is K times more expensive than the ordinary pricing formula.

Fourier methods under uncertainty

$$\tilde{C}^{\theta}(T,k) = \int C^{\theta}(T,k)q(\theta)\mathrm{d}\theta$$
$$= \frac{e^{-\alpha k}}{\pi}\Re\left[\int\int_0^{\infty} g(u)\varphi_T^{\theta}(u-(\alpha+1)i)\, q(\theta)\right]\mathrm{d}u\mathrm{d}\theta,$$
$$= \frac{e^{-\alpha k}}{\pi}\Re\left[\int_0^{\infty} g(u)\tilde{\varphi}^{\theta}(u)\right]\mathrm{d}u \qquad (9)$$

with

$$\tilde{\varphi}_T^{\theta}(u) := \int \varphi_T^{\theta}(u-(\alpha+1)i)\, q(\theta)\mathrm{d}\theta \qquad (10)$$

Closed form solution possible?

Exponentially affine parameters

Assume that the parameters and latent states $\theta = [\gamma\ \beta]$ can be partitioned into two disjoint groups γ and β such that the characteristic function is exponentially affine in β

$$\begin{aligned}\varphi_T^{\theta}(v) &= \exp\left[A^{\gamma}(T,v) + \sum_{l=1}^{p} B_l^{\gamma}(T,v)\beta_l\right] \\ &= \exp\left[A^{\gamma}(T,v) + \mathbf{B}_{\gamma}^{\top}(T,v)\beta\right], \qquad (11)\end{aligned}$$

where $A^{\gamma}(T,v)$ and $B_l^{\gamma}(T,v)$ are known functions that only depends on γ.

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where $A^{\gamma}(T, v)$ and $B_l^{\gamma}(T, v)$ are known functions that only depends on γ.

Example Black & Scholes. The characteristic function is given by (here $\beta = \sigma^2$)

$$\varphi_T^{\theta}(u) = \exp\left[iu(s_0 + rT) + (-i\tfrac{T}{2}u - \tfrac{T}{2}u^2)\sigma^2\right] \tag{12}$$

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Main computational result

- Recall the moment generating function for β,
  $M^{\beta}(w) = \mathbf{E}^{\mathbb{Q}}\left[e^{w\beta}\right] = \int e^{w\beta} q(\beta) \mathrm{d}\beta$

Main computational result

- Recall the moment generating function for β, $M^{\beta}(w) = \mathbf{E}^{\mathbb{Q}}\left[e^{w\beta}\right] = \int e^{w\beta} q(\beta) \mathrm{d}\beta$
- It then follows (using $v = u - (\alpha + 1)i$) that

$$\begin{aligned}\tilde{\varphi}_T^{\theta}(u) &= \int \varphi_T^{\theta}(v)\, q(\beta) \mathrm{d}\beta \\ &= \exp\left[A^{\gamma}\left(T, u - (\alpha+1)i\right)\right] M^{\beta}\left(\mathbf{B}_{\gamma}\left(T, u - (\alpha+1)i\right)\right)\end{aligned}$$

Remarks

- This can be applied for a large class of models (B&S, Merton, Heston, Bates, Exp Levy process, Time Shifted Exp Levy process) etc.
- Can also be extended to time varying parameters
- Some cases result in new exp. affine functions, e.g. Gamma distributed that develops into the generalized beta prime distribution
- Can use framework to distinguish between suitable (e.g uniform) and unsuitable (e.g. lognormal) distributions

Simulations, Bates model

Combination of Heston Stochastic volatility and Merton Jump diffusion model

$$dS_t = \tilde{r} S_t \mathrm{d}t + \sqrt{V_t} S_t \mathrm{d}W_t^{(1)} + S_{t-} dZ_t \qquad (13)$$

$$dV_t = \kappa \left(\Theta - V_t\right) \mathrm{d}t + \eta \sqrt{V_t} \mathrm{d}W_t^{(2)} \qquad (14)$$

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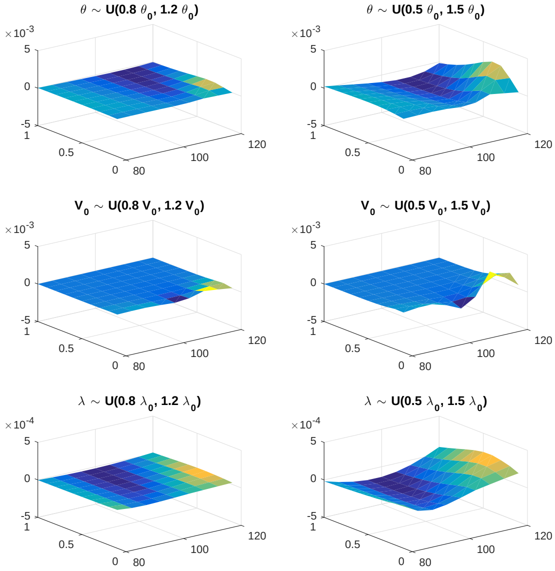
$$\mathrm{d}V_t = \kappa\left(\Theta - V_t\right)\mathrm{d}t + \eta\sqrt{V_t}\mathrm{d}W_t^{(2)} \tag{14}$$

The characteristic function is exponentially affine in the initial volatility, V_0, the long term volatility, Θ, and the jump intensity, λ

$$\varphi_T^{\theta}(u) = \exp\left[A(T,u) + B_{\Theta}(T,u)\Theta + B_{V_0}(T,u)V_0 + B_{\lambda}(T,u)\lambda\right], \tag{15}$$

Difference in implied volatility



Empirical results

Measure error as

$$Q_t = \frac{1}{N} \sum_{i=1}^{N_t} \frac{(\pi_{t,i} - \hat{\pi}_{t,i}(\theta))^2}{(Ask_{t,i} - Bid_{t,i})^2} \tag{16}$$

Interpretation: Errors smaller than unity indicates that the model is good (enough?)

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Two empirical studies on S%P 500 data.

- **Interpolation** use 60 % of data each day for estimation, rest for validation
- **Forecasting** fit using 50 days of data today, evaluate tomorrow with the S_{t+1} known.

Interpolation, Black & Scholes

	In-sample		Out-of-sample	
Model	Mean	Median	Mean	Median
B&S	3.2681	3.1325	3.3347	3.1620
B&S $\Gamma(\sigma^2)$	3.0302	2.8998	3.1525	2.9947
B&S $\Gamma(\Gamma(\sigma^2))$	3.0356	2.9064	3.1571	3.0084
B&S $\sigma^2(t)$	2.8959	2.8811	2.9706	2.9823
B&S $\Gamma(\sigma^2(t))$	1.1724	1.0298	1.2671	1.0667

Interpolation, Merton

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Model	Mean	Median	Mean	Median
B&S	3.2681	3.1325	3.3347	3.1620
B&S $\Gamma(\sigma^2(t))$	1.1724	1.0298	1.2671	1.0667
Merton	1.8566	1.3922	1.9694	1.4379
Merton $\Gamma(\sigma^2)$	1.6397	1.2364	1.7818	1.3077
Merton $\Gamma(\lambda)$	1.5846	1.1863	1.6792	1.2283
Merton $\Gamma(\Gamma(\lambda))$	1.5776	1.1906	1.6725	1.2272
Merton $\sigma^2(t)$	1.5084	1.1009	1.5771	1.1515
Merton $\Gamma(\sigma^2(t))$	0.9336	0.5817	1.0286	0.6349
Merton $\Gamma(\sigma^2(t),\ \lambda)$	0.8886	0.7738	0.9804	0.9249

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Merton $\Gamma(\sigma^2(t),\ \lambda)$	0.8886	0.7738	0.9804	0.9249
Heston	0.4008	0.3428	0.4488	0.3876
Heston $\Gamma(V_0)$	0.3824	0.3297	0.4363	0.3671
Heston $\Gamma(\Theta)$	0.3577	0.3070	0.4080	0.3528

Interpolation, Bates

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Heston	0.4008	0.3428	0.4488	0.3876
Heston $\Gamma(\Theta)$	0.3577	0.3070	0.4080	0.3528
Bates	0.2866	0.2412	0.3342	0.2750
Bates $\Gamma(V_0)$	0.2728	0.2430	0.3189	0.2845
Bates $\Gamma(\Theta)$	0.2461	0.2117	0.2893	0.2459
Bates $\Gamma(\lambda)$	0.2312	0.1978	0.2702	**0.2263**
Bates $\Gamma(\Theta, \lambda)$	**0.2263**	**0.1964**	**0.2667**	0.2271

Forecasting, Black & Scholes

	In-sample		Out-of-sample	
Model	Mean	Median	Mean	Median
BS	3.2956	3.0633	3.5915	3.2825
BS $\Gamma(\sigma)$	3.0861	2.8900	3.4344	3.1278
BS $\Gamma(\Gamma(\sigma^2))$	3.0873	2.8917	3.4353	3.1312
BS $\Gamma(\sigma^2(t))$	1.2714	1.1428	1.9074	1.6112

Forecasting, Merton

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BS	3.2956	3.0633	3.5915	3.2825
BS $\Gamma(\sigma^2(t))$	1.2714	1.1428	1.9074	1.6112
Merton	2.0215	1.5401	2.5162	2.1699
Merton $\Gamma(\sigma^2)$	1.8147	1.3456	2.3335	1.9903
Merton $\Gamma(\lambda)$	1.7255	1.2991	2.2539	1.9485
Merton $\Gamma(\Gamma(\lambda))$	1.7194	1.2996	2.2510	1.9497
Merton $\sigma^2(t)$	1.6331	1.2123	2.1829	1.8811
Merton $\Gamma(\sigma^2(t))$	1.0745	0.7828	1.8139	1.5238
Merton $\Gamma(\sigma^2(t), \lambda)$	0.9169	0.6698	1.6601	1.3187

Forecasting, Heston

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Heston	0.3744	0.3070	1.4453	1.0216
Heston $\Gamma(V_0)$	0.3653	0.3058	1.4348	1.0327
Heston $\Gamma(\Theta)$	0.3483	0.3000	1.4545	1.0177

Forecasting, Bates

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Heston $\Gamma(V_0)$	0.3653	0.3058	1.4348	1.0327
Bates	0.2591	0.2241	1.4143	1.0019
Bates $\Gamma(\Theta)$	0.2538	0.2178	1.3820	**0.9803**
Bates $\Gamma(\lambda)$	0.2338	0.2057	1.3874	0.9893
Bates $\Gamma(\Theta, \lambda)$	**0.2303**	**0.2020**	**1.3718**	0.9848

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Summary & conclusions

- The commonly assumed filtrations are too rich, and we demonstrated how to correct for this.
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Thank you for the attention!

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